Deriving the Evolution of the Ensemble of Galaxies,
or
Quasistatic Equilibria and Stochasticity,

or

Tinsley & Larson (1978) + Mandelbrot and van Ness (1968) + White & Rees (1978)

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2016 March

Something Something about Reionization Something Something

What are people thinking?

- Incomplete knowledge leads us to adopt oversimplified galaxy growth histories

- Galaxy growth assumed to track halo growth explicitly, or

- Toy models used as a substitute for physical understanding, and while

- There are empirical constraints on average histories, is any of it meaningful?

- Sims also too dependent on unresolved physics and don't get equilibria quite right

What do we know?

- Tinsley & Larson (1978), Efstathiou (2000),etc: galaxies in steady-state between inflows, outflows, feedback

- Quasi-static equlibria $\longrightarrow \mathrm{E}[\Delta dM/dt] = 0$

- MCLT allows us to derive: $\mathrm{E}[dM/dt]$, $\mathrm{E}[M]$, $\mathrm{E}[d\ln M/dt]$, and $\mathrm{Sig}[d\ln M/dt]$

- Power spectrum gives us $1/f$ noise in the growth of structure, correlating the things that happen over time, producing fractional Gaussian noise in the stochastic histories of equilibrium states

Let us begin: SFR occurs as an Equilibrium Process

Given a sequence of stellar mass growths at interval t, $S_0, S_1, S_2, \ldots, S_{t+1}$, let us define X_{t+1}*,

$$X_{t+1} = S_{t+1} - S_t$$

In other words,

$$S_t = (S_t - S_{t-1}) + (S_{t-1} - S_{t-2}) + (S_{t-2} - S_{t-3}) + \cdots + S_0$$

$$S_t = \sum_{i=1}^{t} X_i$$

And remember that

$$M_{t+1} = \sum_{i=1}^{t} S_i$$

$$M_{t+1} = \sum_{i=1}^{t} \sum_{j=1}^{i} X_j$$

* Warning: astrophysics buried here.

Let us begin: SFR occurs as an Equilibrium Process

So SFR vs M is apparently just a correlation between $\sum_i^t X_i$ and $\sum_i^t \sum_j^i X_j$.

Can we work out how those two sums should be correlated?

S is stationary, so $E[X] = 0$. But there is a variance σ_t^{2*}:

$$\mathrm{Var}[S_t - S_{t-1}] = \sigma_t^2$$

Note: σ_t's need not reflect any Gaussian or Gaussian-like distribution in the X_t's.

Only need X_t's to be bounded.

Believe it or not, we are now ready to say a lot about how representative ensembles of galaxies evolve over cosmic time.

* Astrophysics buried here!

The Martingale Central Limit Theorem

If the stochastic differences, X, are i.r.v. centered on zero, then S is called a "martingale," and X are "martingale differences."

Why do you care about this?

Sums of sequences of such numbers obey central limit theorems.

If you have central limit theorems, you can compute probabilities and expectation values!

The Martingale Central Limit Theorem

We need to compute the variance in S_t:

$$\mathrm{Var}[S_t] = E[S_t^2] - (E[S_t])^2$$

Given that S is stationary, centered on $S_0 = 0$, $E[S_t] = 0$, and thus

$$\mathrm{Var}[S_t] = \sum_{i=1}^{t} X_i^2 = \sum_{i=1}^{t} \sigma_i^2$$

where σ_i is the expected variance in the stochastic changes to S at time i.

Let's take an ensemble of N object histories $S_{n,t}$, where $n \in \{1, 2, 3, \ldots, N\}$.

Each object, n, has a history, with different variances at every timestep, etc.

We therefore define an RMS stochastic fluctuation for n's history up to $S_{n,t}$:

$$\overline{\sigma}_{n,t} = \left(\frac{1}{t}\sum_{i=1}^{t} \sigma_{n,i}^2\right)^{1/2}$$

The Martingale Central Limit Theorem

Note these RMS stochastic fluctuations for each n history up to time t,

$$\overline{\sigma}_{n,t} = \left(\frac{1}{t}\sum_{i=1}^{t}\sigma_{n,i}^{2}\right)^{1/2}$$

have all the macro-physics that changes galaxy equilibrium states.

All the micro-physics is subsumed into the fact that star formation occurs as an equilibrium process.

The central limit theorem states that the distribution of $S_{n,t}$, normalized by these RMS fluctuations,

$$\frac{S_{n,t}}{t^{1/2}\overline{\sigma}_{n,t}} = \frac{1}{t^{1/2}\overline{\sigma}_{n,t}}\sum_{i=1}^{t} X_{n,i}$$

is a Gaussian centered in zero with a standard deviation of unity:

$$\frac{S_{n,t}}{t^{1/2}\overline{\sigma}_{n,t}} \xrightarrow{d} N(0,1)$$

Imposing Nonnegativity

Stellar mass growth is almost always nonnegative.

Imposing $S \geq 0$ turns S into a submartingale, and S, on average tends to go up.
Every submartingale can be expressed as the sum of:

(1) a martingale (yay!), and

(2) a long-term drift term

The resulting limit for $S \geq 0$ is the nonnegative half of the Gaussian:

$$P\left[\frac{S_{n,t}}{t^{1/2}\overline{\sigma}_{n,t}} < x\right] = \left(\frac{2}{\pi}\right)^{1/2} \int_0^x e^{-x^2/2} dx$$

WE NOW HAVE A PROBABILITY DISTRIBUTION.

Let us now skip doing the integrals and just write down the 1st and 2nd moments.

Markovian Expectation Values

So far we have derived a probability distribution for S_t assuming the timesteps are independent of each other for each galaxy.

In other words galaxies at time t don't care what they've done previously.

You get 1st and 2nd moments of dP/dx, plus the integral of the 1st moment:

$$E\left[\frac{S_t}{\overline{\sigma}}\right] = \sqrt{\frac{2}{\pi}} t^{1/2}$$

$$\mathrm{Var}\left[\frac{S_t}{\overline{\sigma}}\right] = \frac{1}{2} E\left[\frac{S_t}{\overline{\sigma}}\right]^2$$

$$E\left[\frac{M_t}{\overline{\sigma}}\right] = \frac{2}{3}\sqrt{\frac{2}{\pi}} t^{3/2}$$

A Markovian Star-Forming Main Sequence

If galaxies grow in a sort of steady-state, with stochastic changes to their growth rates, and every stochastic change to a galaxy's growth rate is independent of the other stochastic changes in its history, one gets this SFMS:

$$E\left[\frac{S_t}{M_t}\right] = \frac{3}{2t}$$

$$\mathrm{Sig}\left[\frac{S_t}{M_t}\right] = \frac{1}{\sqrt{2}} E\left[\frac{S_t}{M_t}\right]$$

$$\mathrm{Sig}\left[\ln \frac{S_t}{M_t}\right] \approx \frac{1}{\sqrt{2}}$$

$$\mathrm{Sig}\left[\log \frac{S_t}{M_t}\right] \approx 0.3\ \mathrm{dex}$$

The bad news is that galaxies aren't Markovian.

Covariant Stochasticity: Timesteps are Correlated

In reality, galaxy n's history has long- and short-term correlations between stochastic changes to its growth:

$$Y_{n,t} = \sum_{j=0}^{m} c_{n,t,t-j} X_{n,t-j}$$

$$S_{n,t} = \sum_{i=1}^{t} Y_{n,i}$$

There is an unknown, seemingly unconstrained set of covariances between stochastic changes in S.

Guess what: sums of m-dependent random variables also obey limit theorems!

Furthermore, $1/f$ noise (i.e. $P(k) = k^{-1}$) leads to fractional Gaussian noise and well understood distributions for Y.

Covariant Stochasticity: Convergence in Distribution

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Convergence of weighted sums of random variables with long-range dependence ☆

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Abstract

Suppose that f is a deterministic function, $\{\xi_n\}_{n\in\mathbb{Z}}$ is a sequence of random variables with long-range dependence and B_H is a fractional Brownian motion (fBm) with index $H\in(\frac{1}{2},1)$. In this work, we provide sufficient conditions for the convergence

$$\frac{1}{m^H}\sum_{n=-\infty}^{\infty} f\left(\frac{n}{m}\right)\xi_n \to \int_{\mathbb{R}} f(u)\,\mathrm{d}B_H(u)$$

in distribution, as $m\to\infty$. We also consider two examples. In contrast to the case when the ξ_n's are i.i.d. with finite variance, the limit is not fBm if f is the kernel of the Weierstrass–Mandelbrot process. If however, f is the kernel function from the "moving average" representation of a fBm with index H', then the limit is a fBm with index $H+H'-\frac{1}{2}$. © 2000 Published by Elsevier Science B.V.

Covariant Stochasticity: fractional Brownian motion

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FRACTIONAL BROWNIAN MOTIONS, FRACTIONAL NOISES AND APPLICATIONS*

BENOIT B. MANDELBROT† AND JOHN W. VAN NESS‡

1. Introduction. By "fractional Brownian motions" (fBm's), we propose to designate a family of Gaussian random functions defined as follows:[1] $B(t)$ being ordinary Brownian motion, and H a parameter satisfying $0 < H < 1$, fBm of exponent H is a moving average of $dB\,(t)$, in which past increments of $B(t)$ are weighted by the kernel $(t - s)^{H-1/2}$. We believe fBm's do provide useful models for a host of natural time series and wish therefore to present their curious properties to scientists, engineers and statisticians.

The basic feature of fBm's is that the "span of interdependence" between their increments can be said to be infinite. By way of contrast, the study of random functions has been overwhelmingly devoted to sequences of independent random variables, to Markov processes, and to other random functions having the property that sufficiently distant samples of these functions are independent, or nearly so. Empirical studies of random chance phenomena often suggest, on the contrary, a strong interdependence between distant samples. One class of examples arose in economics. It is known that economic time series "typically" exhibit cycles of all orders of magnitude, the slowest cycles having periods of duration comparable to the total sample size. The sample spectra of such series show no sharp "pure period" but a spectral density with a sharp peak near frequencies close to the inverse of the sample size [1], [4]. Another class of examples arose in the study of fluctuations in solids. Many such fluctuations are called "1:*f* noises," because their sample spectral density takes the form λ^{1-2H}, with λ the frequency, $\frac{1}{2} < H < 1$ and H frequently close to 1. Since, however, values of H far from 1 are also frequently observed, the term "1:*f* noise" is inaccurate. It is also unwieldy. With some trepidation due to the availability of

Covariant Stochasticity: fractional Brownian motion

2. The definition of fractional Brownian motion. As usual, t designates time, $-\infty < t < \infty$, and ω designates the set of all the values of a random function. (This ω belongs to a sample space Ω.) The ordinary Brownian motion, $B(t, \omega)$, of Bachelier, Wiener and Lévy is a real random function with independent Gaussian increments such that $B(t_2, \omega) - B(t_1, \omega)$ has mean zero and variance $|t_2 - t_1|$, and such that $B(t_2, \omega) - B(t_1, \omega)$ is independent of $B(t_4, \omega) - B(t_3, \omega)$ if the intervals (t_1, t_2) and (t_3, t_4) do not overlap. The fact that the standard deviation of the increment $B(t + T, \omega) - B(t, \omega)$, with $T > 0$, is equal to $T^{1/2}$ is often referred to as the "$T^{1/2}$ law."

DEFINITION 2.1. Let H be such that $0 < H < 1$, and let b_0 be an arbitrary real number. We call the following random function $B_H(t, \omega)$, *reduced fractional Brownian motion* with parameter H and starting value b_0 at time 0. For $t > 0$, $B_H(t, \omega)$ is defined by

$$B_H(0, \omega) = b_0,$$

$$B_H(t, \omega) - B_H(0, \omega)$$

$$= \frac{1}{\Gamma(H + \frac{1}{2})} \left\{ \int_{-\infty}^{0} [(t - s)^{H-1/2} - (-s)^{H-1/2}] \, dB(s, \omega) + \int_0^t (t - s)^{H-1/2} \, dB(s, \omega) \right\} \tag{2.1}$$

fractional Brownian motion: the long and the short of it

We already derived the case for non-negative Brownian histories.

The fBm models are generalizations governed by the Hurst parameter: $0 \leq H \leq 1$.

When $H = 0.5$ timesteps are all independent (Brownian). When $H < 0.5$ there is antipersistence. When $H > 0.5$ there is persistence (positive feedback).
Technically the bounds are not inclusive, because when $H = 1$ the integral only converges at $t = \infty$.

For such a case, it would be like the universe was a system with an ensemble of galaxies that never forgot their pasts.

Nonnegative fBm: Expectation Values

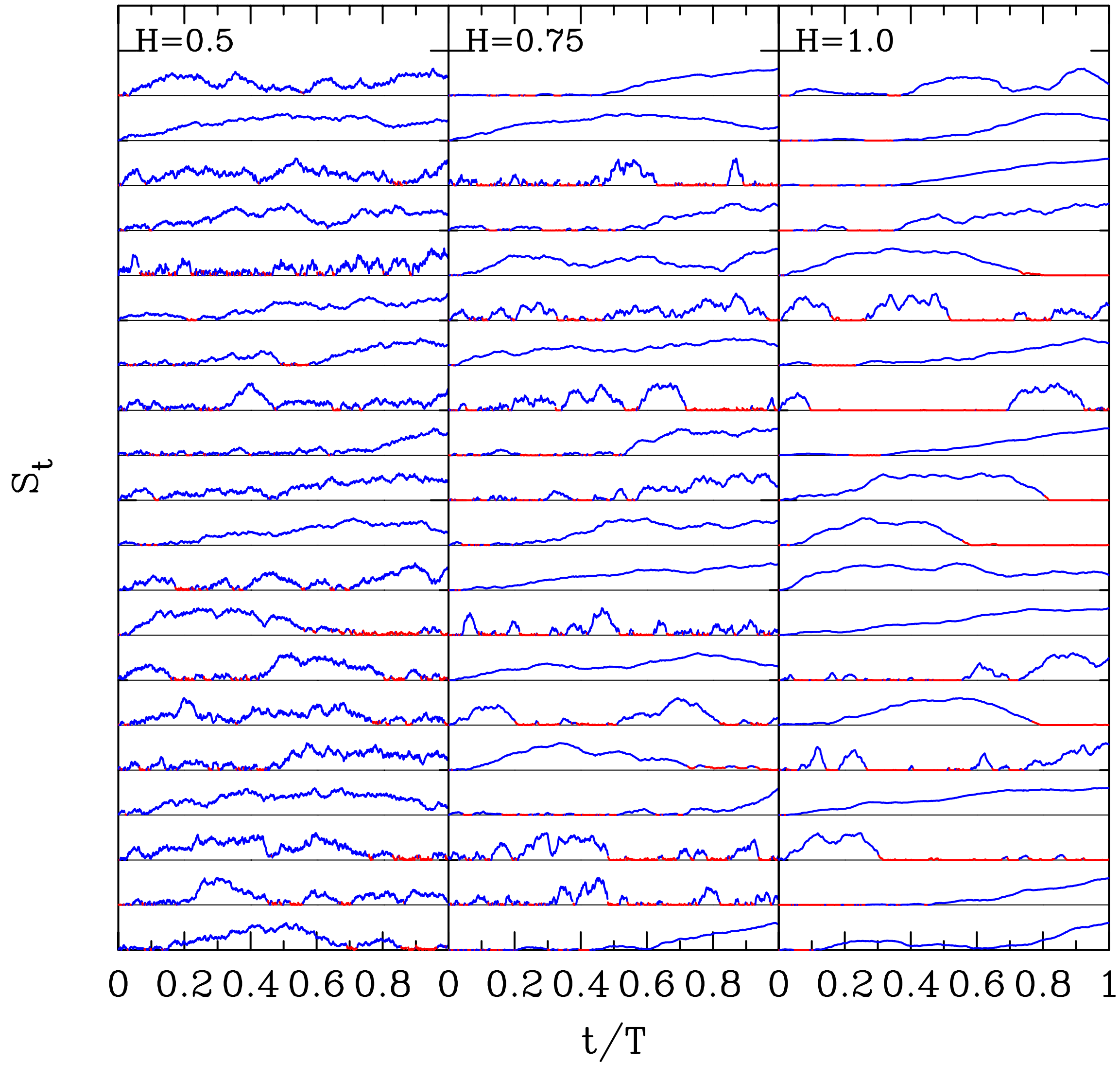
Serving both the interests of the audience and the speaker, let us just jump to:

$$E[S_t] = \overline{\sigma}\frac{1}{\sqrt{2\pi}}\left(\frac{t^H}{H}\right)$$

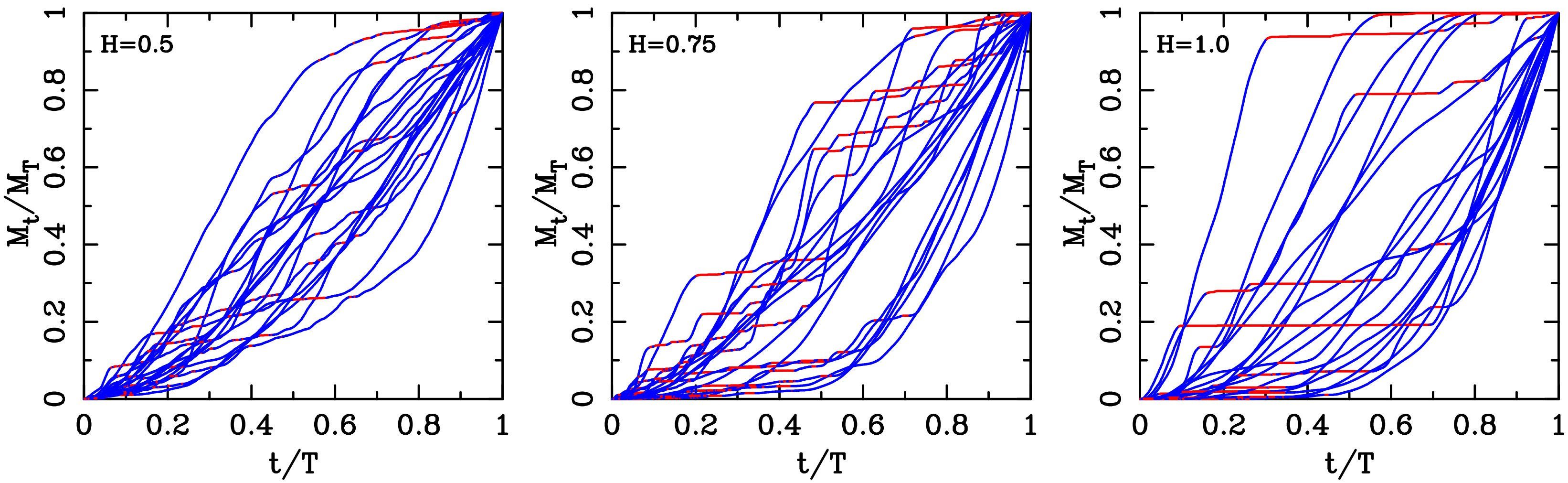
$$\mathrm{Sig}[S_t] = H^{1/2}E[S_t]$$

$$E[M_t] = \overline{\sigma}\frac{1}{\sqrt{2\pi}}\left[\frac{t^{H+1}}{(1+H)H}\right]$$

Nonnegative fBm: Example Scale-Free Histories



Nonnegative fBm: Example Scale-Free Growth Histories



Nonnegative fBm: The Star-Forming Main Sequence

The expectation values for S_t and M_t, again, are both proportional to $\overline{\sigma}$.

Thus one obtains a generalized SFMS of:

$$E[S_t/M_t] = \frac{H+1}{t}$$

$$\mathrm{Sig}[S_t/M_t] = H^{1/2} E[S_t/M_t]$$

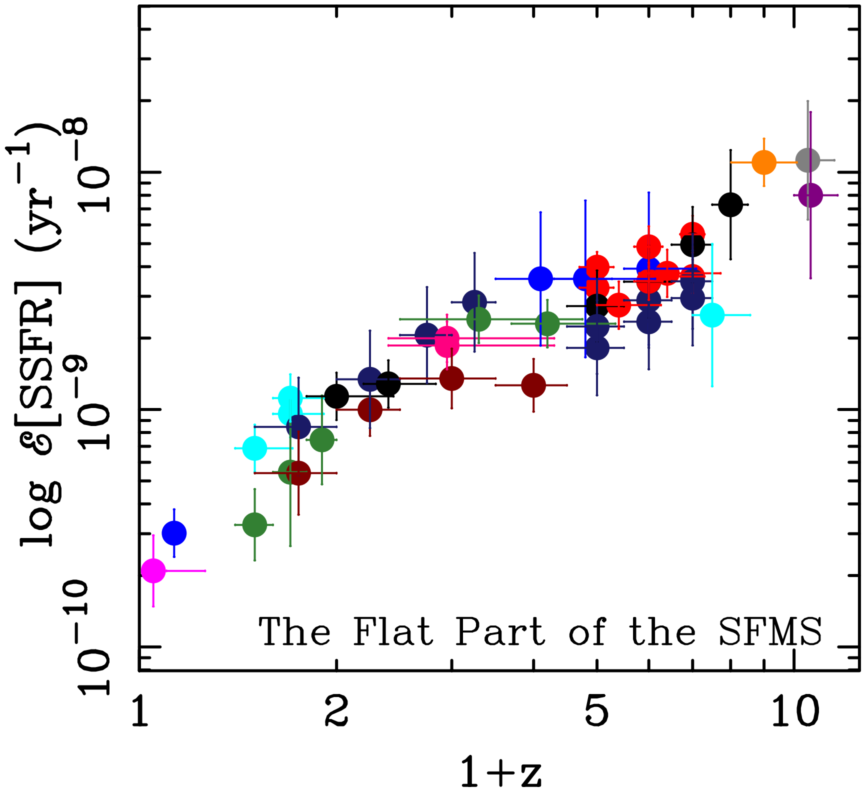
These expectations are valid for representative ensembles, so watch out for your selection effects!

Different kinds of changes in long-term macrophysics will affect these expectations, e.g. massive galaxies at late times.

IOW this is the flat part of the SFMS.

Back to the Star-Forming Main Sequence

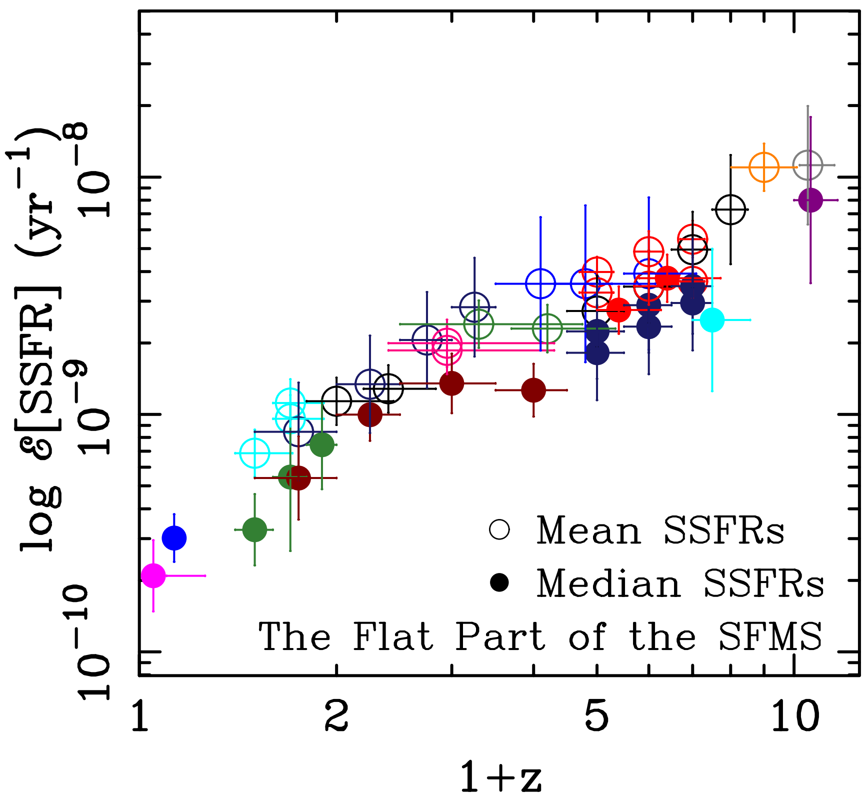
Lots of data from the literature for the flat part of the SFMS, selecting those samples deep enough to not be biased against passive galaxies.



These data look like a fracking mess. How would when even begin to test whether the predictions are correct?

Rethinking those Measurements

Turns out that different people measure different things, artificially inflating the apparent disagreement among datasets.



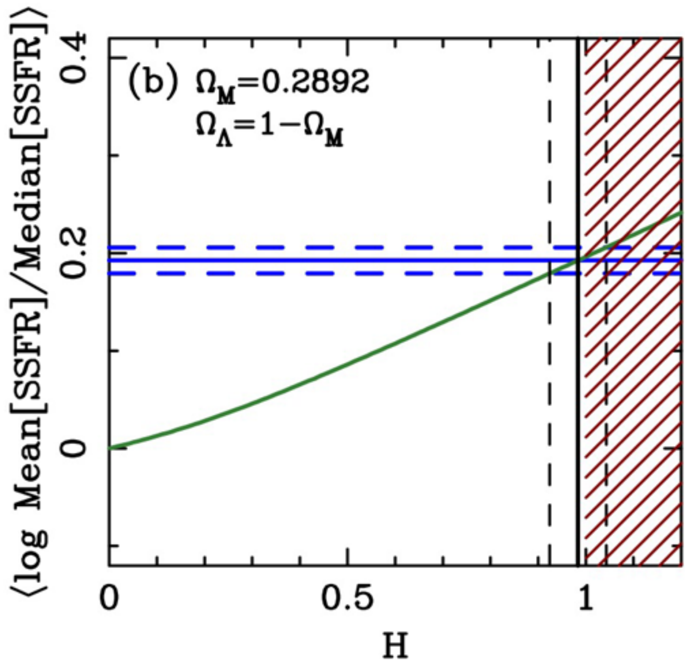
The difference between a mean and a median

Recall that

$$E[S_t/M_t] = \frac{H+1}{t}$$
$$\mathrm{Sig}[S_t/M_t] = H^{1/2} E[S_t/M_t]$$

This scatter translates directly to an offset between the mean and median SSFR.

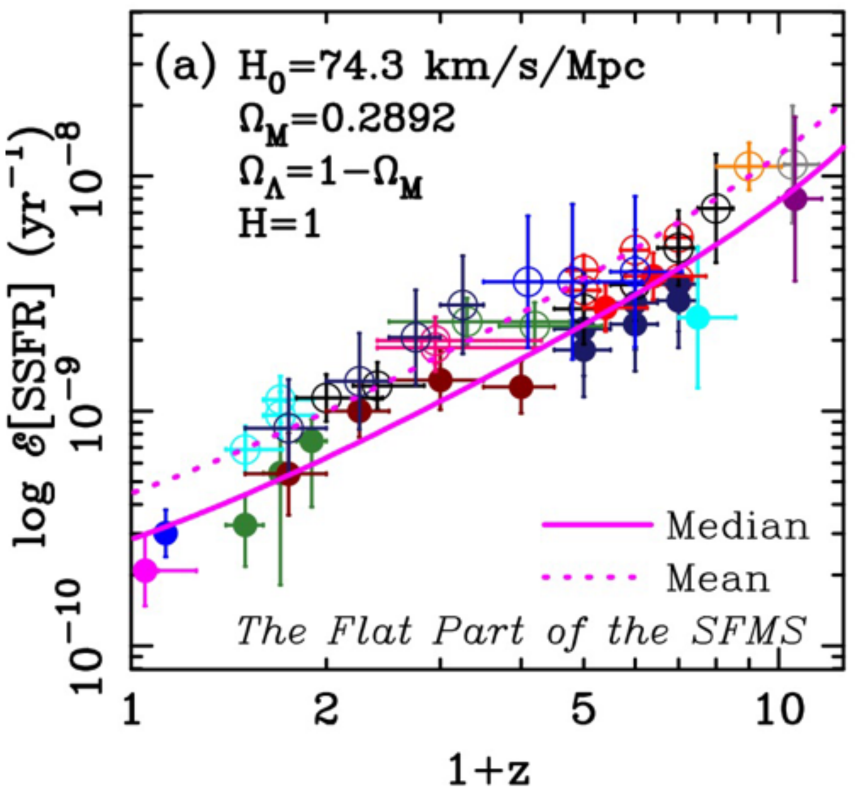
Let's fit A/t to the mean SSFRs and B/t to the medians and compute $\log A/B$:



So galaxies are a bit like elephants

Here the violet solid line is the predicted locus for Median[SSFR] vs redshift.

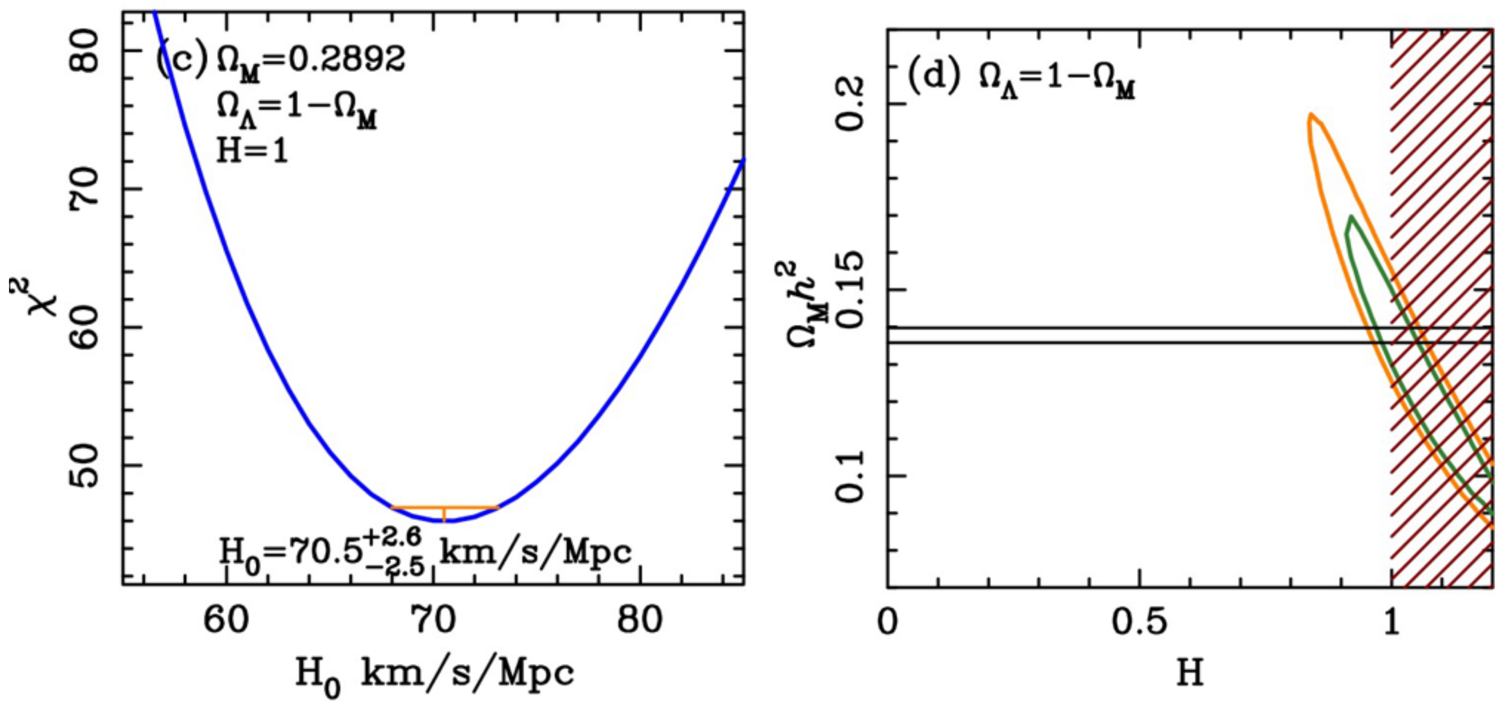
The violet dashed line is the predicted locus for the Mean[SSFR] vs redshift.



There's no way this is an accident

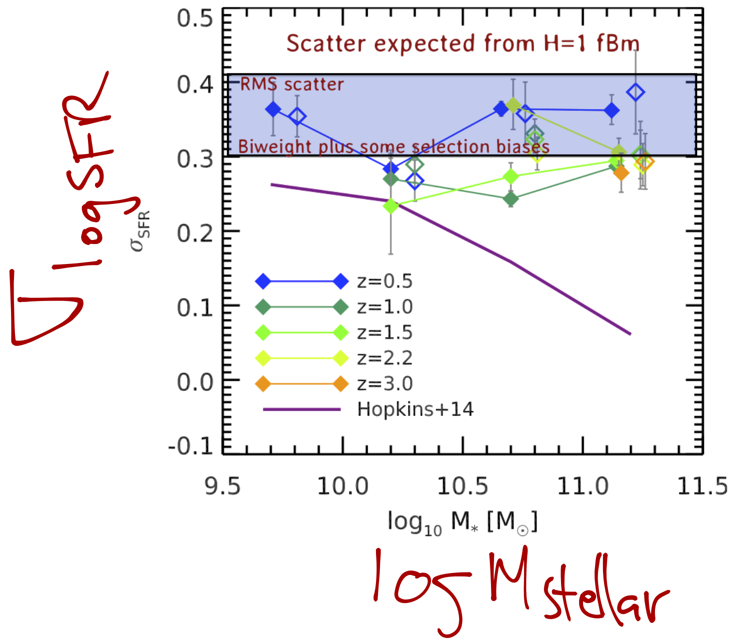
We derived that the Median[S/M] on the flat-part of the SFMS is identically $2/t$.

The implication is that *every* published Median[S/M] is therefore a cosmic clock.



IOW: $2/t$ goes right through the medians, and $2/t \times 1.57$ right through the means. To a few pct.

What about the expectation value for the scatter?



Schreiber et al (2015)

Salim et al (2007): 0.4 dex intrinsic @low-z. Gonzalez et al (2014): ~0.5 dex @hi-z.

Very difficult to measure right; selection biases matter a lot.

Do not measure for SF gals only! Samples must be cosmologically representative!

But if you wondered why the scatter in SSFR appears to (relatively) independent of mass and time. This is why.

Let Us Breath And Quickly Take Stock

- The SFMS is emergent, *not* deterministic
- The SFMS does not imply that more massive galaxies form stars at greater rates!
- Rather: in order for a galaxy of mass M to have formed by z, it had to have formed stars on average more vigorously than lower mass galaxies.
- The set of SFHs implied by fBm is quite diverse (and infinite).
- Implied histories show activity and inactivity on a range of timescales (feasts and famines)

Galaxy Evolution at Early Times

The lack of dependence of SSFR on M at early times implies that we know the ensemble of SFHs for galaxies at high-z

Except we have some seemingly arbitrary scale factor out front:

$$E[M_t] = \overline{\sigma} \frac{t^2}{2\sqrt{2\pi}}$$

In the SFMS $\overline{\sigma}$ was just a nuisance, something we could ignore because it cancelled out.

But $\overline{\sigma}$ normalizes the SFRs and stellar masses, and is thus critical for computing stellar mass functions over time!

Can we calculate $\overline{\sigma}$ a priori?

A Characteristic Stochastic Fluctuation Amplitude

Let us start with

$$E\left[\frac{dM}{dt}\right] = \frac{\overline{\sigma}}{\sqrt{2\pi}}t$$

Let us then take the first derivative (investigate ensembles for which the RMS fluctuation is roughly constant over some time interval):

$$\frac{d}{dt}E\left[\frac{dM}{dt}\right] = \frac{\overline{\sigma}}{\sqrt{2\pi}}$$

$$E\left[\frac{d^2M}{dt^2}\right] = \frac{\overline{\sigma}}{\sqrt{2\pi}}$$

Let us simplify dM/dt as the rate of accretion of baryons, converted to stars with some fraction ϵ, where v_b is the infall velocity and ρ_b is the ambient density:

$$\frac{dM}{dt} = \epsilon\rho_b v_b$$

A Characteristic Stochastic Fluctuation Amplitude

We'll use a simple top-hat approximation, and other assumptions about the density of the ambient medium being relatively constant over a short enough timescale at the start of the stochastic process S, so that:

$$\frac{d^2M}{dt^2} = \epsilon\rho_b\frac{dv_b}{dt}$$

$$= \epsilon\rho_b\frac{GM_h}{R_h^2}$$

which eventually will look like

$$\frac{d^2M}{dt^2} = \epsilon f_b\left(\frac{4\pi 178}{3}\right)^{2/3} GM_h^{1/3}\rho^{5/3}$$

$$\overline{\sigma}^* = \sqrt{2\pi}\epsilon f_b\left(\frac{4\pi 178}{3}\right)^{2/3} GM_h^{1/3}\rho^{5/3}$$

Using characteristic halo mass at the onset of star-formation, and the matter density at that epoch, one has the characteristic mass scaling.

Note: weak dependence on M_h, and very strong dependence on environment!

A Characteristic Stochastic Fluctuation Amplitude

Popular halo mass functions for $z \sim 10$ have characteristic $M_h \sim 6 \times 10^9 M_\odot$ (e.g. Warren et al 2006, Tinker et al 2008).

Let us adopt a rate of conversion of baryons to stars of 2%, and baryon fraction $f_b = 0.15$.

This number is what goes in front of, e.g., $E[M_t] = \overline{\sigma} t^2/(2\sqrt{2\pi})$:

$$\overline{\sigma}^* \approx \left(\frac{\epsilon}{0.02}\right)\left(\frac{f_b}{0.15}\right)\left(\frac{1+z}{1+10}\right)^{7/3}\left(\frac{M_h}{7.5 \times 10^9 M_\odot}\right)^{1/3} \times \left(1.1 \times 10^{-7} M_\odot/\mathrm{yr}^2\right)$$

(this formula also takes into account the evolution in characteristic M_h with redshift, so all one need do is change z to the epoch when you think SF starts)

The Spectrum of Stochastic Fluctuation Amplitudes

Note that $\overline{\sigma}$ has a dependence on both halo mass and local density:

$$\overline{\sigma} \propto M_h^{1/3} \rho^{5/3}$$

The exponent of 1/3 on M_h means the halo mass function itself does not dominate the shape of the $P(\overline{\sigma})$ at low $\overline{\sigma}$.

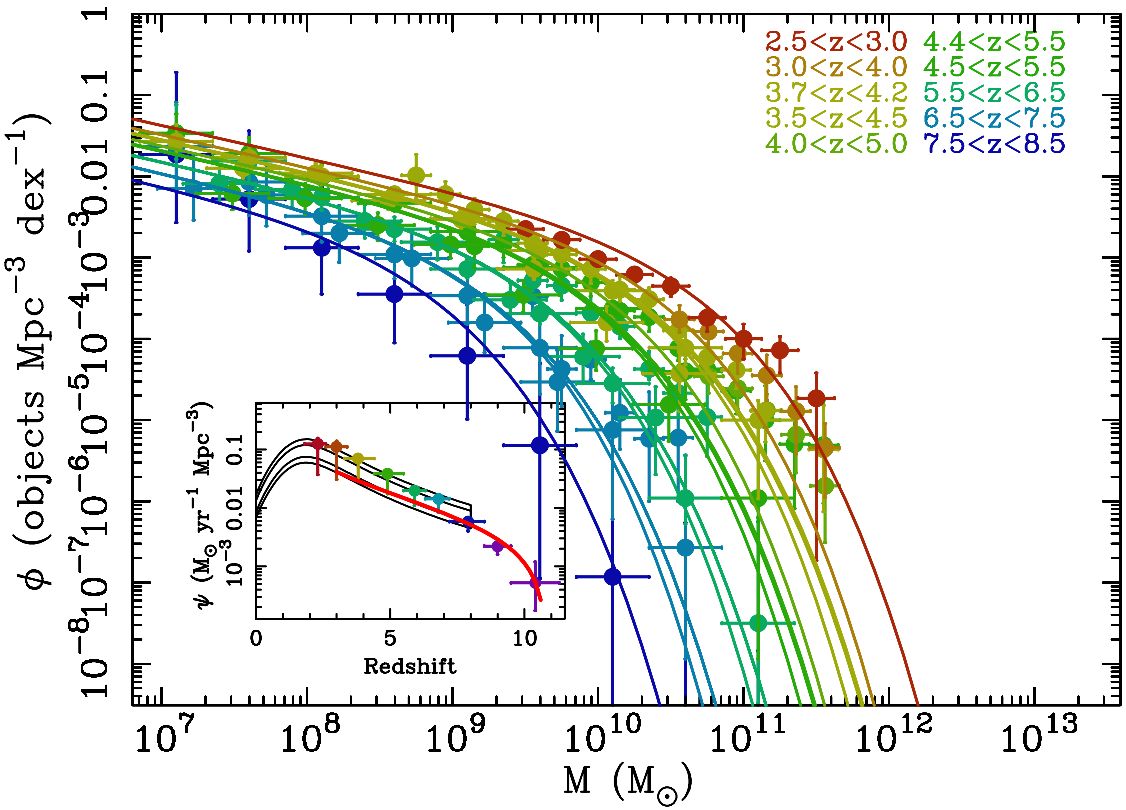
On the physical scales that drive our stochastic changes to S, recall that $P(k) \sim k^{-3}$. This means $P(\rho) \sim \rho^{-5/3}$. And thus

$$P(\overline{\sigma}) \sim \overline{\sigma}^{\ -7/5}$$

Because $M \propto \overline{\sigma} t^2$, this P-L slope implies an $\alpha = -7/5$ for MF (at, e.g., fixed z_{start})

Simplest Model for Early Times

Let's use this characteristic SF acceleration and define a spectrum of values for galaxy seeds, and...

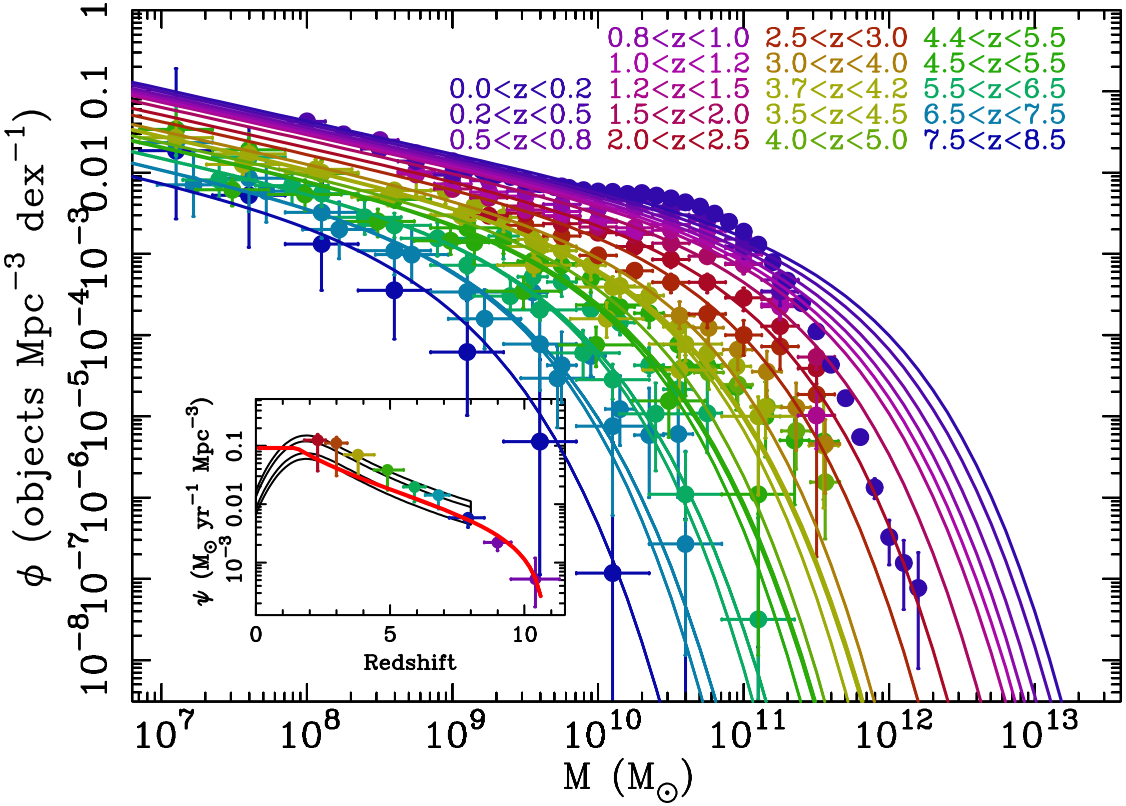


(SFRD from Madau & Dickinson 2014; mass functions from various.)

This simple model adopted a single z_start=10.8. This may be a valid average, but the sum over 10<z_start<20 ought to be better.

Simplest Model Evolved to Late Times

Let's use this characteristic SF acceleration and define a spectrum of values for galaxy seeds, and...

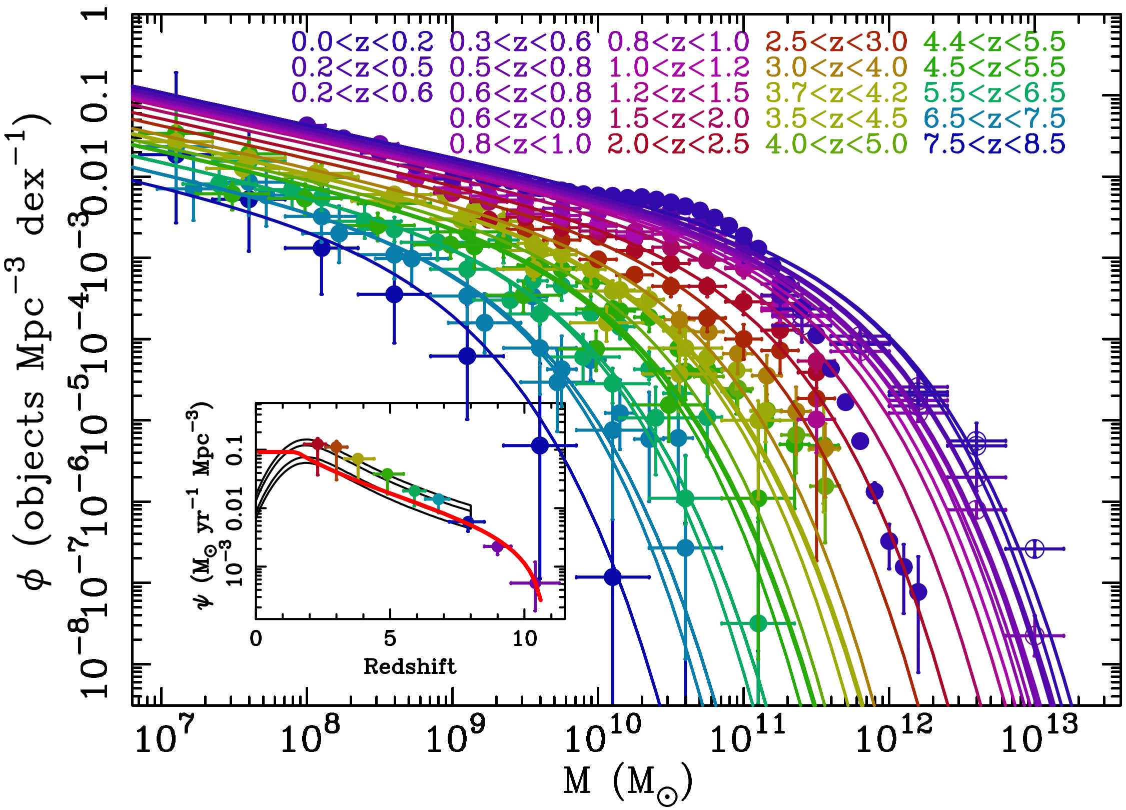


(SFRD from Madau & Dickinson 2014; mass functions from various.)

Hey look, it totally messes up on massive galaxies at late times. We need a second process. I wonder what it could be?

Simplest Model Evolved to Late Times

Let's use this characteristic SF acceleration and define a spectrum of values for galaxy seeds, and...



(SFRD from Madau & Dickinson 2014; mass functions from various.)

Oh, I see, it has something to do with the accretion of galaxies into groups. Since that's the CSI group stellar mass function over there.

Some of the Relevant Points to Take Away

- The "Star-Forming Main Sequence" is emergent, a natural consequence of stellar mass growth as a stochastic process
- The non-Markovian-ness arises naturally from the $1/f$ noise in the power spectrum
- Derive $\mathrm{E}[(dM/dt)/M] = 2/t$, accurately matching SSFRs on flat part of SFMS over $0<z<10$
- Observed intrinsic scatter in SSFR at fixed mass falls right out
- Stellar mass functions and Madau diagram $3 \lesssim z \lesssim 10$
- Infinite set of possible SFHs, including those of local group dwarf gals, MW
- Retrodict quiescent galaxy fractions along flat part of SFMS
- Strongly limits how well one can link specific progenitors with specific descendents
- Average histories really, really are not indicative of what individual galaxies do
- Must trace full ensembles over cosmic time, but we now have math to help us!
- This framework is almost complete — have almost worked out what happens at late times
- Very simply explains rising SFHs for early galaxies.
- Leads to characteristic mass growth as t^2; naturally produces characteristic stellar mass scales at all epochs, up to galaxy group scales at present